FACTORIAL SUPERSYMMETRIC SCHUR FUNCTIONS
AND SUPER CAPELLI IDENTITIES

ALEXANDER MOLEV

Centre for Mathematics and its Applications,
Australian National University,
Canberra, ACT 0200, Australia
(e-mail: molev@pell.anu.edu.au)

Abstract

A factorial analogue of the supersymmetric Schur functions is introduced. It is shown that factorial versions of the Jacobi–Trudi and Sergeev–Pragacz formulae hold. The results are applied to construct a linear basis in the center of the universal enveloping algebra for the Lie superalgebra $\mathfrak{gl}(m|n)$ and to obtain super-analogues of the higher Capelli identities.

Mathematics Subject Classifications (1991). 05A15, 05E05, 17A70

0. Introduction

For a partition λ of length $\leq m$ the factorial Schur function $s_\lambda(x|a)$ in variables $x = (x_1, \ldots, x_m)$ depending on an arbitrary numerical sequence $a = (a_i)$, $i \in \mathbb{Z}$ may be defined as follows [20, p. 54]. Let

$$(x|a)^k = (x - a_1) \cdots (x - a_k)$$

for each $k \geq 0$. Then

$$s_\lambda(x|a) = \frac{\det \left[(x_j|a)^{\lambda_i + m - i} \right]_{1 \leq i,j \leq m}}{\Delta(x)}, \tag{0.1}$$

where $\Delta(x)$ stands for the Vandermonde determinant,

$$\Delta(x) = \prod_{i<j}(x_i - x_j) = \det \left[(x_j|a)^{m-i} \right]_{1 \leq i,j \leq m}.$$

Note that the usual Schur function $s_\lambda(x)$ coincides with $s_\lambda(x|a)$ for the zero sequence a. The factorial Schur functions admit many of the classical properties of $s_\lambda(x)$ (see [4–6, 9, 10, 21, 29, 31, 32]), as well as some new ones, e.g., the characterization theorem [29] which plays an important role in the proofs of analogues of the Capelli identity [24, 29]. (The term 'factorial' was primarily used only for the case of the sequence a with $a_i = i - 1$; we use it in the present paper in a broader sense, for an arbitrary a). In particular, the functions $s_\lambda(x|a)$ may be equivalently defined in terms of tableaux which also enables one to introduce the skew factorial Schur functions $s_{\lambda/\mu}(x|a)$ for each pair of partitions $\mu \subset \lambda$.

Super-analogues of the Schur functions can be defined by using some specializations of the usual symmetric functions in infinitely many variables (see [20, p. 58]). This approach goes back to Littlewood [19], see also [23]. On the other hand, they naturally emerged in the representation theory of Lie superalgebras [15, 16] and were studied by several authors, see, e.g., [2, 3, 7, 9, 13, 33–35, 38–40].

For a pair of partitions λ and μ with $\mu \subset \lambda$ the supersymmetric skew Schur function $s_{\lambda/\mu}(x/y)$ in variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ can be defined by the formula

$$s_{\lambda/\mu}(x/y) = \sum_{\mu \subset \nu \subset \lambda} s_{\lambda/\nu}(x) s_{\nu'/\mu'}(y), \tag{0.2}$$

where λ' denotes the partition conjugate to λ.

Linear span of the functions (0.2) consists of all supersymmetric polynomials P in x and y; that is, those polynomials which are symmetric in x and y separately and satisfy the following cancellation property: the result of setting $x_m = -y_n = z$ in P is independent of z.

In this paper we introduce factorial analogues $s_{\lambda/\mu}(x/y\,|a)$ of the supersymmetric Schur functions parametrized by arbitrary numerical sequences $a = (a_i)$, $i \in \mathbb{Z}$ (see Section 1). For the zero sequence a this function coincides with $s_{\lambda/\mu}(x/y)$,

the space of supersymmetric polynomials. We prove that many properties of the supersymmetric Schur functions have their factorial analogues.

First, we find the generating series for the corresponding elementary and complete functions and check that they are supersymmetric.

Then, using a modified Gessel–Viennot method [8] we prove an analogue of the Jacobi-Trudi formula and thus prove that the polynomials $s_{\lambda/\mu}(x/y\,|a)$ are also supersymmetric.

Further, we prove a characterization theorem for the polynomials $s_\lambda(x/y\,|a)$ analogous to the corresponding theorem for the factorial Schur polynomials [29] (see also [37]).

Using this theorem we prove a factorial analogue of the Sergeev–Pragacz formula. For the usual supersymmetric Schur functions this formula can be proved by several different ways, see, e.g., [3, 13, 22, 33, 34]. However, these proofs can not be easily carried to the case of the functions $s_\lambda(x/y\,|a)$, because for a general sequence a they lose both the symmetry property of the functions (0.2)

$$s_{\lambda/\mu}(x/y) = s_{\lambda'/\mu'}(y/x) \tag{0.3}$$

and the specialization property with respect to x

$$s_\lambda(x/y)|_{x_m=0} = s_\lambda(x'/y), \tag{0.4}$$

where $x' = (x_1, \ldots, x_{m-1})$.

In particular, in the case of $a = (0)$ we obtain one more proof of the Sergeev–Pragacz formula.

As a corollary of the factorial Sergeev–Pragacz formula we get an analogue of the Berele–Regev factorization theorem for the polynomials $s_\lambda(x/y\,|a)$ [2], see also [9, 35].

A special case of the factorization theorem yields an analogue of the dual Cauchy formula which gives a decomposition of the double product $\prod\prod(x_i + y_j)$ into a sum of products of the factorial Schur functions. Two other proofs of this formula are given in [17, 18] and [21].

Goulden and Greene [9] and Macdonald [21] found a new tableau representation for the functions $s_{\lambda/\mu}(x/y)$ in the case of infinite sets of variables $x = (x_i)$ and $y = (y_i)$, $i \in \mathbb{Z}$. Using this representation we prove that the corresponding function $s_{\lambda/\mu}(x/y\,|a)$ does not depend on a (here $a = (a_i)$ is regarded as a sequence of independent variables) and coincides with $s_{\lambda/\mu}(x/y)$.

Finally, we show that many results of the papers [28–31] concerning higher Capelli identities and shifted Schur functions have their natural super-analogues. In particular, a basis in the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n))$ is explicitly constructed. The eigenvalues of the basis elements in highest weight representations are super-analogues of the shifted Schur functions. We outline two proofs of the super-versions of the higher Capelli identities. The first proof uses the characterization theorem for the factorial supersymmetric Schur polynomials while the second one is based on the properties of the Jucys–Murphy elements in the group algebra for the symmetric group. These identities include the super Capelli identity found by Nazarov [27].

I would like to thank A. Lascoux, M. Nazarov, A. Okounkov, G. Olshanskiĭ, P.

1. Definition and combinatorial interpretation of the functions $s_{\lambda/\mu}(x/y\,|a)$

We shall suppose that $a = (a_i)$, $i \in \mathbb{Z}$ is a fixed sequence of complex numbers.

For a pair of partitions $\mu \subset \lambda$ the skew factorial Schur function can be defined by the formula (see, e.g., [9, 21]):

$$s_{\lambda/\mu}(x|a) = \sum_{T} \prod_{\alpha \in \lambda/\mu} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}), \qquad (1.1)$$

summed over all semistandard skew tableaux T of shape λ/μ with entries in the set $\{1, \ldots, m\}$, where $T(\alpha)$ is the entry of T in the cell α and $c(\alpha) = j - i$ is the content of $\alpha = (i, j)$. The entries of a semistandard tableau are supposed to be weakly increasing along rows and strictly increasing down columns. It can be verified directly (see [9]) that the polynomials (1.1) are symmetric in x. It was proved in [21] that for a standard (non-skew) shape λ formulae (0.1) and (1.1) define the same function, which makes the symmetry property obvious for this case.

Due to (1.1), the factorial elementary and complete symmetric polynomials are given by

$$e_k(x|a) = s_{(1^k)}(x|a) = \sum_{i_1 < \cdots < i_k} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2-1}) \cdots (x_{i_k} - a_{i_k-k+1}), \quad (1.2)$$

$$h_k(x|a) = s_{(k)}(x|a) = \sum_{i_1 \leq \cdots \leq i_k} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2+1}) \cdots (x_{i_k} - a_{i_k+k-1}). \quad (1.3)$$

It is immediate from (1.1) that the highest component of $s_{\lambda/\mu}(x|a)$ is the usual skew Schur polynomial $s_{\lambda/\mu}(x)$. This implies that the polynomials $s_\lambda(x|a)$ with $l(\lambda) \leq m$ form a basis in the symmetric polynomials in x.

Definition 1.1. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be two families of variables. Given a sequence a denote by $a^* = (a_i^*)$ another sequence, defined by $a_i^* = -a_{n-i+1}$, and introduce the *factorial supersymmetric Schur polynomials (functions)* by the formula:

$$s_{\lambda/\mu}(x/y\,|a) = \sum_{\mu \subset \nu \subset \lambda} s_{\lambda/\nu}(x|a) s_{\nu'/\mu'}(y|a^*). \qquad (1.4)$$

We shall prove in Section 3 that these polynomials are indeed supersymmetric, so their name will be justified.

Comparing (0.2) and (1.4) we see that the highest homogeneous component of the polynomial $s_{\lambda/\mu}(x/y\,|a)$ is $s_{\lambda/\mu}(x/y)$ if the latter is nonzero, and that $s_{\lambda/\mu}(x/y\,|a)$ coincides with $s_{\lambda/\mu}(x/y)$ for the zero sequence a.

Using formula (1.1) we can reformulate definition (1.4) in terms of tableaux. To distinguish the indices of x and y let us identify the indices of y with the symbols $1', \ldots, n'$. Consider the diagram of shape λ/μ and fill it with the indices $1', \ldots, n', 1, \ldots, m$ such that:

(a) In each row (resp. column) each primed index is to the left (resp. above) from each unprimed index.

(b) Primed indices strictly decrease along rows and weakly decrease down columns.

(c) Unprimed indices weakly increase along rows and strictly increase down columns.

Denote the resulting tableau by T.

Proposition 1.2. *One has the formula*

$$s_{\lambda/\mu}(x/y\,|a) = \sum_T \prod_{\substack{\alpha\in\lambda/\mu \\ T(\alpha)\,\text{unprimed}}} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}) \prod_{\substack{\alpha\in\lambda/\mu \\ T(\alpha)\,\text{primed}}} (y_{T(\alpha)} + a_{T(\alpha)+c(\alpha)}).$$

$$(1.5)$$

Proof. For each tableau T the cells of the diagram λ/μ occupied by primed indices form a subdiagram ν/μ where $\mu\subset\nu\subset\lambda$. Let us fix such a partition ν and sum in (1.5) first over the tableaux T whose primed part forms a subtableau of shape ν/μ. The part of such a tableau T formed by unprimed indices is a semistandard subtableau of shape λ/ν and taking the sum over these subtableaux we get by (1.1) that

$$\sum_T \prod_{\alpha\in\lambda/\nu} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}) = s_{\lambda/\nu}(x|a).$$

It remains to verify that

$$\sum_T \prod_{\alpha\in\nu/\mu} (y_{T(\alpha)} + a_{T(\alpha)+c(\alpha)}) = s_{\nu'/\mu'}(y|a^*), \qquad (1.6)$$

summed over ν/μ-tableaux T with entries from $\{1,\ldots,n\}$ whose rows strictly decrease and columns weakly decrease. Indeed, since $s_{\nu'/\mu'}(y|a^*)$ is symmetric in y, setting $\widetilde{y} = (y_n,\ldots,y_1)$ and using (1.1) we may write

$$s_{\nu'/\mu'}(y|a^*) = s_{\nu'/\mu'}(\widetilde{y}\,|a^*) = \sum_{T'} \prod_{\alpha'\in\nu'/\mu'} (\widetilde{y}_{T'(\alpha')} - a^*_{T'(\alpha')+c(\alpha')})$$

$$= \sum_{T'} \prod_{\alpha'\in\nu'/\mu'} (y_{n-T'(\alpha')+1} + a_{n-T'(\alpha')+1-c(\alpha')}), \qquad (1.7)$$

summed over semistandard ν'/μ'-tableaux T' with entries from $\{1,\ldots,n\}$. Note that the map

$$T'(\alpha') \to T(\alpha) = n - T'(\alpha') + 1,$$

where $\alpha = (i,j) \in \nu/\mu$ and $\alpha' = (j,i) \in \nu'/\mu'$, is a bijection between the set of semistandard ν'/μ'-tableaux and the set of ν/μ-tableaux whose rows strictly decrease and columns weakly decrease. Obviously, $c(\alpha) = -c(\alpha')$, hence, (1.7) coincides with the left hand side of (1.6) which completes the proof.

2. Generating series for the elementary and complete factorial supersymmetric polynomials

Introduce now the *elementary* and *complete factorial supersymmetric polynomials* as special cases of $s_\lambda(x/y\,|a)$ with λ being a column or row partition, respectively:

Using definition (1.4) we can express them in terms of the polynomials (1.2) and (1.3) as follows:

$$e_k(x/y\,|a) = \sum_{p+q=k} e_p(x|\tau^{-q}a)h_q(y|a^*), \tag{2.1}$$

$$h_k(x/y\,|a) = \sum_{p+q=k} h_p(x|\tau^q a)e_q(y|a^*), \tag{2.2}$$

where τ is a shift operator acting on sequences a by replacing each a_i by a_{i+1}. Proposition 1.2 yields the following explicit formulae for $e_k(x/y\,|a)$ and $h_k(x/y\,|a)$:

$$e_k(x/y\,|a) = \sum_{p+q=k} \sum_{\substack{i_1<\cdots<i_p \\ j_1\geq\cdots\geq j_q}} (y_{j_1}+a_{j_1})\cdots(y_{j_q}+a_{j_q-q+1})(x_{i_1}-a_{i_1-q})\cdots(x_{i_p}-a_{i_p-k+1}),$$
$$\tag{2.3}$$

$$h_k(x/y\,|a) = \sum_{p+q=k} \sum_{\substack{i_1\leq\cdots\leq i_p \\ j_1>\cdots>j_q}} (y_{j_1}+a_{j_1})\cdots(y_{j_q}+a_{j_q+q-1})(x_{i_1}-a_{i_1+q})\cdots(x_{i_p}-a_{i_p+k-1}).$$
$$\tag{2.4}$$

We shall suppose that $e_k(x/y\,|a) = h_k(x/y\,|a) = 0$ if $k < 0$. Note that formulae (2.3) and (2.4) imply the following symmetry property:

$$h_k(x/y\,|a) = e_k(y/x\,|-\tau^{k-1}a).$$

Theorem 2.1. *One has the following generating series for the polynomials $e_k(x/y\,|a)$ and $h_k(x/y\,|a)$:*

$$1+\sum_{k=1}^{\infty} \frac{(-1)^k e_k(x/y\,|a)}{(t-a_{m-k+1})\cdots(t-a_m)} = \frac{(t-x_1)\cdots(t-x_m)(t-a_1)\cdots(t-a_n)}{(t-a_1)\cdots(t-a_m)(t+y_1)\cdots(t+y_n)}, \tag{2.5}$$

$$1+\sum_{k=1}^{\infty} \frac{h_k(x/y\,|a)}{(t-a_{m+1})\cdots(t-a_{m+k})} = \frac{(t-a_1)\cdots(t-a_m)(t+y_1)\cdots(t+y_n)}{(t-x_1)\cdots(t-x_m)(t-a_1)\cdots(t-a_n)}. \tag{2.6}$$

Proof. We shall use induction on m. The generating series for the complete factorial symmetric polynomials $h_k(y|a)$ is given by

$$1 + \sum_{k=1}^{\infty} \frac{h_k(y|a)}{(t-a_{n+1})\cdots(t-a_{n+k})} = \frac{(t-a_1)\cdots(t-a_n)}{(t-y_1)\cdots(t-y_n)}. \tag{2.7}$$

This formula was proved in [31, 32] in the special case of the sequence a with $a_i = i - 1$ and this proof works in the general case as well. Note that for $m = 0$ we have $e_k(x/y\,|a) = h_k(y|a^*)$ by (2.1), and (2.5) follows from (2.7). Suppose now that $m \geq 1$. Denote $x' = (x_1, \ldots, x_{m-1})$. We see from (2.3) that

$$e_k(x/y\,|a) = e_k(x'/y\,|a) + e_{k-1}(x'/y\,|a)(x_m - a_{m-k+1}). \tag{2.8}$$

So, using the induction hypotheses, we may write the right hand side of (2.5) as

$$\sum^{\infty} \frac{(-1)^k e_k(x'/y\,|a)}{(t-a\ )\cdots(t-a\ )} \cdot \frac{t-x_m}{t-a}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k e_k(x'/y\,|a)}{(t - a_{m-k+1}) \cdots (t - a_m)} \cdot \frac{t - x_m}{t - a_{m-k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k e_k(x'/y\,|a) + (-1)^k e_{k-1}(x'/y\,|a)(x_m - a_{m-k+1})}{(t - a_{m-k+1}) \cdots (t - a_m)},$$

where we have used

$$\frac{t - x_m}{t - a_{m-k}} = 1 - \frac{x_m - a_{m-k}}{t - a_{m-k}}.$$

Due to (2.8) this proves (2.5).

The following formula for the generating series for the elementary factorial symmetric polynomials $e_k(y|a)$ is contained in [20, p. 55] (see also [31, 32]):

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k e_k(y|a)}{(t - a_{n-k+1}) \cdots (t - a_n)} = \frac{(t - y_1) \cdots (t - y_n)}{(t - a_1) \cdots (t - a_n)}. \tag{2.9}$$

For $m = 0$ we have $h_k(x/y\,|a) = e_k(y|a^*)$ by (2.2) and so, for this case (2.6) follows from (2.9). Now let $m \geq 1$. We see from (2.4) that

$$h_k(x/y\,|a) = \sum_{r+s=k} h_r(x'/y\,|a)(x_m - a_{m+k-s}) \cdots (x_m - a_{m+k-1}). \tag{2.10}$$

By the induction hypotheses we can write the right hand side of (2.6) in the form

$$\sum_{r=0}^{\infty} \frac{h_r(x'/y\,|a)}{(t - a_m) \cdots (t - a_{m+r-1})} \cdot \frac{t - a_m}{t - x_m}$$

$$= \sum_{r=0}^{\infty} \frac{h_r(x'/y\,|a)}{(t - a_{m+1}) \cdots (t - a_{m+r})} \cdot \frac{t - a_{m+r}}{t - x_m}$$

$$= \sum_{r=0}^{\infty} \frac{h_r(x'/y\,|a)}{(t - a_{m+1}) \cdots (t - a_{m+r})} \sum_{s=0}^{\infty} \frac{(x_m - a_{m+r}) \cdots (x_m - a_{m+r+s-1})}{(t - a_{m+r+1}) \cdots (t - a_{m+r+s})},$$

where we have applied (2.7) with $n = 1$. The latter expression can be rewritten as

$$\sum_{k=0}^{\infty} \frac{1}{(t - a_{m+1}) \cdots (t - a_{m+k})} \sum_{r+s=k} h_r(x'/y\,|a)(x_m - a_{m+k-s}) \cdots (x_m - a_{m+k-1}),$$

which coincides with the left hand side of (2.6) by (2.10).

Corollary 2.2. *For any k the polynomials $e_k(x/y\,|a)$ and $h_k(x/y\,|a)$ are supersymmetric.*

Proof. Indeed, the cancellation property is obviously satisfied by $e_k(x/y\,|a)$ and $h_k(x/y\,|a)$ because after setting $x_m = -y_n = z$ the factors on the right hand sides of (2.5) and (2.6) containing z cancel.

Remark. In his letter to the author A. Lascoux pointed out that the factorial Schur functions are recovered as a special case of the double Schubert polynomials [18]

[22]. In particular, using the technique of divided differences one can prove that the complete factorial supersymmetric polynomials $h_k(x/y\,|a)$ coincide with the complete factorial symmetric polynomials $h_k(x \cup a^{(n)}\,|-y \cup a)$, $a^{(n)} = (a_1, \ldots, a_n)$, and one can also obtain the above generating series for $h_k(x/y\,|a)$ (see also [1]).

3. Jacobi–Trudi formula

The following analogue of the Jacobi–Trudi formula holds for the functions $s_{\lambda/\mu}(x/y\,|a)$.

Theorem 3.1.

$$s_{\lambda/\mu}(x/y\,|a) = \det\left[h_{\lambda_i - \mu_j - i + j}(x/y\,|\tau^{\mu_j - j + 1}a)\right]_{1 \le i,j \le l} \tag{3.1}$$

where $l = l(\lambda)$.

Proof. We use a modified Gessel–Viennot method [8, 36] (cf. [9, 31, 32, 35]). Consider a grid consisting of two parts; the upper half of the grid if formed by m horizontal lines labelled by $1, \ldots, m$ northwards and vertical lines consequently labelled by the elements of $\mathbb{Z}$ eastwards. For each $i \in \mathbb{Z}$ the vertical line labelled by i breaks into two lines at the intersection point with the horizontal line 1. One of the two lines goes south-east and maintain the label i and the other goes south-west and is labelled by i'.

Each vertex of the grid will be denoted by a pair of coordinates (c, i) or (c, i') which are the labels of the lines intersecting at the vertex, so that c labels a vertical line or a line going south-east. We shall consider paths in this grid of the following kind. Each step of a path is north or east in the upper half of the grid and is north-east or north-west in the lower part of the grid. We label each eastern step $(c, i) \to (c + 1, i)$ of a path with $x_i - a_{i+c}$ and each north-eastern step $(c, i') \to (c + 1, i')$ with $y_{i-c} + a_i$. For a path π denote by $L(\pi)$ the product of these labels with respect to all eastern and north-eastern steps. Let us check now that

$$\sum_{\pi} L(\pi) = h_{s-r}(x/y\,|\tau^r a), \tag{3.2}$$

summed over all paths π with the initial vertex $(r, (n + r)')$ and the final vertex (s, m). Indeed, the left hand side of (3.2) can be calculated as follows. First fix a number q such that $0 \le q \le \min\{s - r, n\}$ and find the sum

$$\sum_{\pi_{\text{lower}}} L(\pi_{\text{lower}}), \tag{3.3}$$

where π_{lower} runs over the paths in the lower part of the grid with the initial vertex $(r, (n + r)')$ and the final vertex $(r + q, (r + q)')$ (which belongs to the horizontal line 1 and has also the coordinates $(r + q, 1)$ if regarded as a vertex of the upper half of the grid). Suppose that such a path π_{lower} has north-eastern steps of the form

$$(r, (j_1 + r)') \to (r + 1, (j_1 + r)'),$$

$$\cdots$$

where $n \geq j_1 \geq \cdots \geq j_q \geq q$. Then the product of the labels of π_{lower} with respect to the north-eastern steps equals

$$(y_{j_1} + a_{j_1+r}) \cdots (y_{j_q-q+1} + a_{j_q+r}).$$

Hence, the sum (3.3) equals

$$\sum_{n \geq j_1 \geq \cdots \geq j_q \geq q} (y_{j_1} + a_{j_1+r}) \cdots (y_{j_q-q+1} + a_{j_q+r})$$

$$= \sum_{n \geq i_1 > \cdots > i_q \geq 1} (y_{i_1} + a_{i_1+r}) \cdots (y_{i_q} + a_{i_q+q-1+r}).$$

Applying (1.6) for $\nu/\mu = (q)$ we conclude that this expression coincides with $e_q(y|(\tau^r a)^*)$.

Similarly, one can easily check that the sum

$$\sum_{\pi_{\text{upper}}} L(\pi_{\text{upper}}),$$

where π_{upper} runs over all paths in the upper half of the grid with the initial vertex $(r+q, 1)$ and the final vertex (s, m), coincides with $h_{s-r-q}(x|\tau^{r+q}a)$.

Thus, the left hand side of (3.2) equals

$$\sum_q h_{s-r-q}(x|\tau^{r+q}a) e_q(y|(\tau^r a)^*),$$

which coincides with $h_{s-r}(x/y\,|\tau^r a)$ by (2.2).

A straightforward application of the Gessel–Viennot argument with the use of (3.2) shows that the determinant on the right hand side of (3.1) can be represented as

$$\sum_{\mathcal{P}} L(\pi_1) \cdots L(\pi_l), \tag{3.4}$$

summed over sets of nonintersecting paths $\mathcal{P} = (\pi_1, \ldots, \pi_l)$, where the initial vertex of π_i is $(\mu_i - i + 1, (n + \mu_i - i + 1)')$ and the final vertex is $(\lambda_i - i + 1, m)$.

It remains to prove that (3.4) coincides with $s_{\lambda/\mu}(x/y\,|a)$. Given set $\mathcal{P} = (\pi_1, \ldots, \pi_l)$ of nonintersecting paths we construct a λ/μ-tableau T as follows. Note that the total number of north-eastern and eastern steps of the path π_i equals $\lambda_i - \mu_i$. Let us numerate these steps starting with the initial vertex of π_i. If the kth step is north-eastern of the form $(c, j') \to (c+1, j')$ then the entry of the kth cell in the i-th row of λ/μ is $(j-c)'$ and if the kth step is eastern of the form $(c, j) \to (c+1, j)$ then the entry of this cell is j. It can be easily seen that this correspondence is a bijection between the sets of nonintersecting paths and the tableaux used in the combinatorial interpretation (1.5) of the polynomials $s_{\lambda/\mu}(x/y\,|a)$. Moreover, the corresponding summands in (3.4) and (1.5) are clearly the same, which proves the theorem.

Note that for $n = 0$ formula (3.1) turns into the Jacobi–Trudi formula for the factorial Schur functions $s_{\lambda/\mu}(x|a)$ while for $m = 0$ we get a factorial analogue of the Nägelsbach–Kostka formula (see [20, p. 56]) and setting $a = (0)$ we obtain the Jacobi–Trudi formula for the functions $s_{\lambda/\mu}(x/y)$ [35].

Corollary 2.2 and Theorem 3.1 imply

Corollary 3.2. *The polynomials $s_{\lambda/\mu}(x/y\,|a)$ are supersymmetric.*

An analogue of (3.1) for the factorial elementary supersymmetric polynomials can be proved in the same way by using a grid obtained from the one we have used in the above proof by interchanging its upper and lower parts. We state this formula here without proof.

Theorem 3.3.

$$s_{\lambda'/\mu'}(x/y\,|a) = \det\left[e_{\lambda_i-\mu_j-i+j}(x/y\,|\tau^{-\mu_j+j-1}a)\right]_{1\leq i,j\leq l}. \tag{3.5}$$

4. Characterization theorem

Our aim in this section is to obtain characterization properties for the supersymmetric polynomials (Theorems 4.5 and 4.5′) which will play an important role in our proof of a factorial analogue of the Sergeev–Pragacz formula (Theorem 5.1) and super-analogues of the Capelli identity (Theorem 8.1). We start with investigating vanishing and specialization properties of the polynomials $s_\lambda(x/y\,|a)$.

We say that a partition λ is contained in the (m,n)-hook if $\lambda_{m+1} \leq n$. Note that

$$s_\lambda(x/y\,|a) = 0 \quad \text{unless} \quad \lambda \subset (m,n)\text{-hook}. \tag{4.1}$$

Indeed, by definition (1.4)

$$s_\lambda(x/y\,|a) = \sum_{\rho\subset\lambda} s_{\rho'}(y|a^*)s_{\lambda/\rho}(x|a). \tag{4.2}$$

Clearly, $s_{\rho'}(y|a^*) = 0$ unless $\rho_1 \leq n$. If λ is not contained in the (m,n)-hook then $\lambda'_{n+1} > m$. So, if $\rho_1 \leq n$ then $s_{\lambda/\rho}(x|a) = 0$.

For each $\lambda \subset (m,n)$-hook we introduce two partitions $\mu = \mu(\lambda)$ and $\nu = \nu(\lambda)$ as follows. Nonzero parts of μ are defined by $\mu_i = \lambda_i - n$ for $\lambda_i > n$ and nonzero parts of ν are defined by $\nu_j = \lambda'_j - m$ for $\lambda'_j > m$.

For any partition α such that $l(\alpha) \leq l$ introduce the l-tuple

$$a_\alpha = a_\alpha^{(l)} = (a_{\alpha_1+l}, \ldots, a_{\alpha_l+1})$$

of elements of the sequence a. We shall only consider m- or n-tuples a_α as values for the variables $x = (x_1, \ldots, x_m)$ or $y = (y_1, \ldots, y_n)$ so we shall not specify the number l if it is clear from the context.

We shall often use the following vanishing properties of the factorial Schur polynomials $s_\lambda(x|a)$ (see [29, 31]). For any partition σ such that $l(\sigma) \leq m$ and $\lambda \not\subset \sigma$ one has

$$s_\lambda(a_\sigma|a) = 0. \tag{4.3}$$

Moreover, if $l(\lambda) \leq m$ then

$$s_\lambda(a_\lambda|a) = \prod (a_{\lambda_i+m-i+1} - a_{m-\lambda'_j+j}). \tag{4.4}$$

In particular,

$$s_\lambda(a_\lambda|a) \neq 0,$$

provided that the sequence a is multiplicity free, that is, $a_i \neq a_j$ if $i \neq j$. Relations (4.3) and (4.4) can be easily deduced from either (0.1), or (1.1). In the special case of the sequence a with $a_i = i + \mathrm{const}$ the right hand side of (4.4) turns into the product $H(\lambda)$ of the hook lengths of all cells of λ (see [29]).

Proposition 4.1. *Let η be a partition of length $\leq n$ such that $\nu \not\subset \eta$. Then*

$$s_\lambda(x/a_\eta^* |a) = 0. \tag{4.5}$$

Proof. We use (4.2). Note that $s_{\lambda/\rho}(x|a) = 0$ unless $\lambda_i' - \rho_i' \leq m$ for any i. This means that the sum in (4.2) can be only taken over $\rho \subset \lambda$ such that $\nu' \subset \rho$. By the assumption, $\nu \not\subset \eta$, hence, $\rho' \not\subset \eta$. But in this case $s_{\rho'}(a_\eta^* |a^*) = 0$ by (4.3). Thus, all summands in (4.2) vanish for $y = a_\eta^*$ which completes the proof.

Proposition 4.2. *Let γ be a partition of length $\leq m$ such that $\mu \not\subset \gamma$. Then*

$$s_\lambda((\tau^n a)_\gamma/y\,|a) = 0. \tag{4.6}$$

Proof. We use again that $s_{\rho'}(y|a^*) = 0$ in (4.2) unless $\rho_1 \leq n$. Relation (4.6) will follow from the fact that for any such ρ one has

$$s_{\lambda/\rho}((\tau^n a)_\gamma|a) = 0. \tag{4.7}$$

This relation is a simple generalization of the vanishing property (4.3) and can be proved by similar arguments; see [29, 31]. Indeed, since the left hand side of (4.7) is a polynomial in a, we may assume without loss of generality that the sequence a is multiplicity free. The polynomials $s_{\lambda/\rho}(x|a)$ are symmetric in x, so replacing x with $\widetilde{x} = (x_m, \ldots, x_1)$, we may rewrite definition (1.1) in the following form:

$$s_{\lambda/\rho}(x|a) = \sum_T \prod_{\alpha \in \lambda/\rho} (x_{m-T(\alpha)+1} - a_{T(\alpha)+c(\alpha)}), \tag{4.8}$$

summed over semistandard λ/ρ-tableau T with entries from $\{1, \ldots, m\}$. We shall verify that all summands in (4.8) vanish for $x = (\tau^n a)_\gamma$. Indeed, let us suppose that for some tableau T

$$\prod_{\alpha \in \lambda/\rho} (\{(\tau^n a)_\gamma\}_{m-T(\alpha)+1} - a_{T(\alpha)+c(\alpha)}) \neq 0$$

or equivalently,

$$\prod_{\alpha \in \lambda/\rho} (a_{\gamma_{m-T(\alpha)+1}+T(\alpha)+n} - a_{T(\alpha)+c(\alpha)}) \neq 0.$$

Since a is multiplicity free, this implies that

for all α. For the entries of the first row of the tableau T we have

$$T(1, n+1) \leq \cdots \leq T(1, n + \mu_1). \tag{4.10}$$

Applying (4.9) for $\alpha = (1, n+1)$ we obtain $\gamma_{m-T(1,n+1)+1} \geq 1$. Further, by (4.10),

$$\gamma_{m-T(1,n+2)+1} \geq \gamma_{m-T(1,n+1)+1} \geq 1.$$

By (4.9), applied for $\alpha = (1, n+2)$ we then have $\gamma_{m-T(1,n+2)+1} \geq 2$. Similarly, using an easy induction argument we see that for any $i = 1, \ldots, \mu_1$

$$\gamma_{m-T(1,n+i)+1} \geq i.$$

On the other hand, for the entries of the $(n+i)$th column of T we have

$$T(1, n+i) < \cdots < T(\mu_i', n+i).$$

Hence,

$$\gamma_{m-T(\mu_i',n+i)+1} \geq \cdots \geq \gamma_{m-T(1,n+i)+1} \geq i.$$

This means that $\gamma_i' \geq \mu_i'$, and so, $\mu \subset \gamma$ which contradicts to the assumption of the proposition. The proof is complete.

As we pointed out in Introduction, for a general sequence a the polynomials $s_\lambda(x/y\,|a)$ lose the specialization property of $s_\lambda(x/y)$ with respect to x. However, an analogue of this property with respect to y still holds.

Proposition 4.3. *One has*

$$s_\lambda(x/y\,|a)|_{y_n=-a_n} = s_\lambda(x/y'\,|a), \tag{4.11}$$

where $y' = (y_1, \ldots, y_{n-1})$.

Proof. This follows from the specialization property of the factorial Schur polynomials (see [31, 32]). Indeed, by (1.6) we have

$$s_{\rho'}(y|a^*) = \sum_T \prod_{\alpha \in \rho} (y_{T(\alpha)} + a_{T(\alpha)+c(\alpha)}), \tag{4.12}$$

summed over ρ-tableaux T with entries from $\{1, \ldots, n\}$ whose rows strictly decrease and columns weakly decrease. If for a tableau T one has $T(1,1) = n$, then since $c(1,1) = 0$ the corresponding summand in (4.12) vanishes for $y_n = -a_n$. So we have the property:

$$s_{\rho'}(y|a^*)|_{y_n=-a_n} = s_{\rho'}(y'|a^{*\prime}),$$

where $a^{*\prime}$ is the sequence $(a_i^{*\prime})$ with $a_i^{*\prime} = -a_{n-i}$. Now (4.11) follows from (4.2).

We can now prove a vanishing theorem for the polynomials $s_\lambda(x/y\,|a)$ (cf. [29, 31]). Let ζ be a partition which is contained in the (m,n)-hook. Introduce two other partitions $\xi = \xi(\zeta)$ and $\eta = \eta(\zeta)$ as follows: $\xi = (\zeta_1, \ldots, \zeta_m)$, and nonzero parts of $\eta = (\eta_1, \ldots, \eta_n)$ are defined by $\eta_i = \zeta_i' - m$, if $\zeta_i' > m$.

In particular, $l(\xi) \leq m$ and $l(\eta) \leq n$ and we may consider the m-tuple a_ξ of elements of the sequence a and the n-tuple a_η^* of elements of the sequence a^*.

Theorem 4.4. *Let λ, ζ be partitions which are contained in the (m,n)-hook.*
  (i) *If $\lambda \not\subset \zeta$ then*

$$s_\lambda(a_\xi/a_\eta^*\,|a) = 0. \tag{4.13}$$

  (ii) *If $\lambda = \zeta$ then*

$$s_\lambda(a_\xi/a_\eta^*\,|a) = \prod_{(i,j)\in\lambda}(a_{\lambda_i+m-i+1} - a_{m-\lambda_j'+j}). \tag{4.14}$$

In particular, if the sequence a is multiplicity free then $s_\lambda(a_\xi/a_\eta^*\,|a) \neq 0$.

Proof. Denote by r the length of the partition η. Then

$$a_\eta^* = (-a_{1-\eta_1}, \ldots, -a_{r-\eta_r}, -a_{r+1}, \ldots, -a_n).$$

By Proposition 4.3, the result of setting $y_i = -a_i$ for $i = r+1, \ldots, n$ in the polynomial $s_\lambda(x/y\,|a)$ is the polynomial $s_\lambda(x/y^{(r)}\,|a)$, where $y^{(r)} = (y_1, \ldots, y_r)$. So, (4.13) and (4.14) can be regarded as relations for the families of variables x and $y^{(r)}$. Moreover, due to (4.1) we can only consider the case when λ is contained in the (r,m)-hook. In other words, we may assume without loss of generality that the length of η is n. In particular, the partition (n^m) is contained in ζ.

Now let us prove (i). By Proposition 4.1 we may assume that $\nu \subset \eta$. Since $(n^m) \subset \xi$ we may write $a_\xi = (\tau^n a)_\gamma$, where the partition γ is defined by $\gamma_i = \xi_i - n$. Hence, if $s_\lambda(a_\xi/a_\eta^*\,|a)$ is nonzero then $\mu \subset \gamma$ by Proposition 4.2. Since $\lambda \subset (m,n)$-hook this implies that $\lambda \subset \zeta$ and (i) is proved.

To prove (ii) we note that since $(n^m) \subset \zeta = \lambda$ we have

$$a_\xi = (\tau^n a)_\mu \qquad \text{and} \qquad a_\eta^* = a_\nu^*. \tag{4.15}$$

As we have noticed in the proof of Proposition 4.1, the sum in (4.2) can be only taken over those partitions $\rho \subset \lambda$ for which $\nu' \subset \rho$. On the other hand, using (4.3) we find that $s_{\rho'}(a_\nu^*\,|a^*) = 0$ unless $\rho' \subset \nu$. Hence, for $y = a_\nu^*$ relation (4.2) turns into

$$s_\lambda(x/a_\nu^*\,|a) = s_\nu(a_\nu^*\,|a^*)\,s_{\lambda/\nu'}(x|a). \tag{4.16}$$

Now let us write the polynomial $s_{\lambda/\nu'}(x|a)$ in terms of tableaux using (1.1). By definition of ν we have $\lambda_i' - \nu_i = m$ for all $i = 1, \ldots n$. Since the tableaux T in (1.1) are column strict, all of them have the same entries in the first n columns, namely, the numbers $1, \ldots, m$ written in each of these columns downwards. On the other hand, the entries of the subdiagram μ can form an arbitrary semistandard μ-tableau. For a cell $(i, n+j) \in \lambda$ we obviously have $c(i, n+j) = n + c(i,j)$, so definition (1.1) for $s_{\lambda/\nu'}(x|a)$ takes now the form

$$s_{\lambda/\nu'}(x|a) = s_\mu(x|\tau^n a)\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - a_{j-\nu_j}).$$

We then obtain from (4.16) that

$$s_\lambda((\tau^n a)_\mu/a_\nu^*\,|a) = s_\nu(a_\nu^*\,|a^*)\,s_\mu((\tau^n a)_\mu|\tau^n a)\prod_{i=1}^{m}\prod_{j=1}^{n}(a_{\mu_i+m+n-i+1} - a_{j-\nu_j}).$$

$$\tag{4.17}$$

Now (4.14) follows from (4.4). The theorem is proved. $\qquad\blacksquare$

Theorem 4.4 implies that supersymmetric polynomials in x and y of degree $\leq k$ are characterized by their values at $x = a_\xi$ and $y = a_\eta^*$, where $\xi = \xi(\zeta)$ and $\eta = \eta(\zeta)$ with $|\zeta| \leq k$. More exactly, we have the following result (cf. [29, 31, 37]).

Theorem 4.5. *Suppose that the sequence a is multiplicity free. Let $f(x/y)$ and $g(x/y)$ be supersymmetric polynomials of degree $\leq k$ such that*

$$f(a_\xi/a_\eta^*) = g(a_\xi/a_\eta^*) \tag{4.18}$$

for any partition $\zeta \subset (m,n)$-hook with $|\zeta| \leq k$. Then $f(x/y) = g(x/y)$.

Proof. It is well-known that the functions $s_\lambda(x/y)$ with $\lambda \subset (m,n)$-hook form a basis in supersymmetric polynomials in x and y (see, e.g., [20, p. 61]). So do the functions $s_\lambda(x/y\,|a)$, because the highest term of $s_\lambda(x/y\,|a)$ is $s_\lambda(x/y)$. Hence, we can write the polynomial $f(x/y) - g(x/y)$ as a linear combination of the $s_\lambda(x/y\,|a)$:

$$f(x/y) - g(x/y) = \sum_\lambda c_\lambda s_\lambda(x/y\,|a).$$

Moreover, since the degree of the polynomial on the left hand side $\leq k$ we may assume that $|\lambda| \leq k$. Introduce any total order on the set of partitions such that $|\lambda| < |\mu|$ implies $\lambda < \mu$. The condition (4.18) yields the following homogeneous system of linear equations on the coefficients c_λ:

$$\sum_\lambda c_\lambda s_\lambda(a_\xi/a_\eta^*\,|a) = 0, \qquad |\lambda|, |\zeta| \leq k.$$

Theorem 4.4 implies that the matrix $(s_\lambda(a_\xi/a_\eta^*\,|a))_{\lambda,\zeta}$ of this system, whose rows and columns are arranged in accordance with this order, is triangular with nonzero diagonal elements. Hence, $c_\lambda \equiv 0$ which proves the theorem.

Theorem 4.5 can be obviously reformulated in the following equivalent form.

Theorem 4.5'. *Suppose that the sequence a is multiplicity free. Let $f(x/y)$ be a supersymmetric polynomial such that*

$$f(x/y) = s_\lambda(x/y) + \text{lower terms}$$

for some partition $\lambda \subset (m,n)$-hook and $f(a_\xi/a_\eta^*) = 0$ for any partition $\zeta \subset (m,n)$-hook with $|\zeta| < |\lambda|$. Then $f(x/y) = s_\lambda(x/y\,|a)$.

5. Factorial Sergeev–Pragacz formula

In this section we apply Theorem 4.5 for the proof of an analogue of the Sergeev–Pragacz formula for the polynomials $s_\lambda(x/y\,|a)$. In particular, for $a = (0)$ we get one more proof of the original formula (cf. [3, 13, 22, 33, 34]).

Suppose a partition λ is contained in the (m,n)-hook. Define the partitions μ and ν as in Section 4 and denote by $\rho = (\rho_1, \ldots, \rho_m)$ the part of λ which is contained in the rectangle (n^m), that is, $\rho_i = \min\{\lambda_i, n\}$.

Theorem 5.1.

$$s_\lambda(x/y\,|a) = \frac{\displaystyle\sum_{\sigma\in S_m\times S_n} \mathrm{sgn}(\sigma)\cdot\sigma\{f_\lambda(x/y\,|a)\}}{\Delta(x)\Delta(y)}, \tag{5.1}$$

where

$$f_\lambda(x/y\,|a) = (x_1|\tau^{\rho_1}a)^{\mu_1+m-1}\cdots(x_m|\tau^{\rho_m}a)^{\mu_m}(y_1|a^*)^{\nu_1+n-1}\cdots(y_n|a^*)^{\nu_n}\prod_{(i,j)\in\rho}(x_i+y_j)\cdot \blacksquare$$

Proof. First of all we note that both sides of (5.1) depend polynomially on a so we may assume without loss of generality that the sequence a is multiplicity free.

Denote the right hand side of (5.1) by $\varphi_\lambda(x/y\,|a)$. To apply Theorem 4.5 we have to verify that this polynomial is supersymmetric and that $s_\lambda(x/y\,|a)$ and $\varphi_\lambda(x/y\,|a)$ have the same values at $x = a_\xi$ and $y = a_\eta^*$ for any $\zeta\subset(m,n)$-hook such that $|\zeta|\leq|\lambda|$.

The polynomials $\varphi_\lambda(x/y\,|a)$ are obviously symmetric in x and y, and so, to prove that they are supersymmetric we only need to check that they satisfy the cancellation property. This can be done exactly in the same way as in the case $a = (0)$ (see e.g., [20, p. 61], [33, 34]).

Let us check now that Propositions 4.1–4.3 hold for the polynomials $\varphi_\lambda(x/y\,|a)$ too. To check (4.5) we represent the numerator of the right hand side of (5.1) in the following form:

$$\sum_{\sigma\in S_m}\mathrm{sgn}(\sigma)\cdot\sigma\{(x_1|\tau^{\rho_1}a)^{\mu_1+m-1}\cdots(x_m|\tau^{\rho_m}a)^{\mu_m}\,g_\lambda(x/y\,|a)\},$$

where
$$g_\lambda(x/y\,|a) = \det\big[(y_j|a^*)^{\nu_i+n-i}(y_j+x_1)\cdots(y_j+x_{\rho_i'})\big]_{1\leq i,j\leq n}.$$

The condition $\nu\not\subset\eta$ means that there exists k such that $\eta_k < \nu_k$. For $y = a_\eta^*$ the factor $(y_j|a^*)^{\nu_i+n-i}$ takes the value

$$((a_\eta^*)_j|a^*)^{\nu_i+n-i} = (a_{\eta_j+n-j+1}^* - a_1^*)\cdots(a_{\eta_j+n-j+1}^* - a_{\nu_i+n-i}^*). \tag{5.2}$$

On the other hand, if $i\leq k\leq j$ then

$$1\leq \eta_j+n-j+1\leq \eta_k+n-k+1\leq \nu_k+n-k\leq \nu_i+n-i.$$

This implies that (5.2) is zero and all the ijth entries of the determinant $g_\lambda(x/a_\eta^*\,|a)$ with $i\leq k\leq j$ are zero and so $g_\lambda(x/a_\eta^*\,|a) = 0$. Since the Vandermonde determinant $\Delta(y)$ does not vanish for $y = a_\eta^*$ this proves the assertion. (This argument is very similar to that used in [29, 31] for the proof of (4.3)).

To check that the polynomials $\varphi_\lambda(x/y\,|a)$ satisfy (4.6) we rewrite the numerator of the right hand side of (5.1) in the form:

$$\sum\mathrm{sgn}(\sigma)\cdot\sigma\{(y_1|a^*)^{\nu_1+n-1}\cdots(y_n|a^*)^{\nu_n}\,h_\lambda(x/y\,|a)\},$$

where

$$h_\lambda(x/y\,|a) = \det\left[(x_j|\tau^{\rho_i}a)^{\mu_i+m-i}(x_j+y_1)\cdots(x_j+y_{\rho_i})\right]_{1\le i,j\le m}.$$

The condition $\mu \not\subset \gamma$ implies that for some k one has $\gamma_k < \mu_k$. In particular, this means that $\rho_k = n$ and hence $\rho_1 = \cdots = \rho_k = n$. Therefore, the ijth entry of the determinant $h_\lambda(x/y\,|a)$ for $i \le k \le j$ has the form

$$(x_j|\tau^n a)^{\mu_i+m-i}(x_j+y_1)\cdots(x_j+y_{\rho_i}).$$

Repeating the previous argument, we conclude that $h_\lambda((\tau^n a)_\gamma/y\,|a) = 0$, which completes the proof.

Let us prove now that

$$\varphi_\lambda(x/y\,|a)|_{y_n=-a_n} = \varphi_\lambda(x/y'\,|a), \tag{5.3}$$

where $y' = (y_1,\ldots,y_{n-1})$ and we define $\varphi_\lambda(x/y\,|a) = 0$ if $\lambda \not\subset (m,n)$-hook. Indeed, since $a_1^* = -a_n$, we obviously have for $\nu_n > 0$ that

$$\varphi_\lambda(x/y\,|a)|_{y_n=-a_n} = 0,$$

and hence (5.3) is true, because λ is not contained in the $(m, n-1)$-hook. So, we can suppose that $\nu_n = 0$. Since $\nu_i + n - i > 0$ for $i = 1,\ldots, n-1$, after setting $y_n = -a_n$ we may restrict the sum in the numerator of the right hand side of (5.1) to the set of permutations $\sigma \in S_m \times S_{n-1}$. Further, it can be easily checked that setting $y_n = -a_n$ in $f_\lambda(x/y\,|a)$, we get

$$f_\lambda(x/y\,|a)|_{y_n=-a_n} = f_\lambda(x/y'\,|a)\,(y_1+a_n)\cdots(y_{n-1}+a_n).$$

On the other hand,

$$\Delta(y)|_{y_n=-a_n} = \Delta(y')\,(y_1+a_n)\cdots(y_{n-1}+a_n).$$

The factor $(y_1+a_n)\cdots(y_{n-1}+a_n)$ is symmetric in y', so removing it in the numerator and denominator we see that the result is $\varphi_\lambda(x/y'\,|a)$ which proves (5.3).

Thus, the properties (4.5), (4.6) and (4.11) are satisfied by the polynomials $\varphi_\lambda(x/y\,|a)$. So, repeating the argument that was used for the proof of statement (i) of Theorem 4.4, we obtain that these polynomials also satisfy (4.13). Hence, for any partition $\zeta \subset (m,n)$-hook such that $|\zeta| \le |\lambda|$ and $\zeta \ne \lambda$ we have

$$s_\lambda(a_\xi/a_\eta^*\,|a) = \varphi_\lambda(a_\xi/a_\eta^*\,|a) = 0.$$

Therefore, to apply Theorem 4.5 to the polynomials $s_\lambda(x/y\,|a)$ and $\varphi_\lambda(x/y\,|a)$ it remains to check that

$$s_\lambda(a_\xi/a_\eta^*\,|a) = \varphi_\lambda(a_\xi/a_\eta^*\,|a)$$

for $\zeta = \lambda$. In this case $\eta = \nu$ and $\xi = \rho + \mu$. Due to the specialization properties (4.11) and (5.3) we may assume that $l(\nu) = n$, which implies that the partition ρ coincides with (n^m). In this case we clearly have (see also Corollary 5.2 below)

$$\varphi_\lambda(x/y\,|a) = s_\nu(y|a^*)s_\mu(x|\tau^n a)\prod_{}^{m}\prod_{}^{n}(x_i+y_j).$$

Setting $x = a_\xi = (\tau^n a)_\mu$ and $y = a_\nu^*$ we see that the result coincides with (4.17) which completes the proof of the theorem.

Remark. As it was noticed in [17], formula (5.1) implies that the polynomials $s_\lambda(x/y|a)$ coincide with the *multi-Schur functions* (see, e.g., [22]) in appropriate variables.

As a corollary of Theorem 5.1 we obtain the following factorization theorem for the polynomials $s_\lambda(x/y|a)$ which turns into the Berele–Regev formula for $a = (0)$ [2] (cf. [9, 35]).

Corollary 5.2. *If a partition λ is contained in the (m,n)-hook and contains the partition (n^m) then*

$$s_\lambda(x/y|a) = s_\mu(x|\tau^n a)s_\nu(y|a^*)\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j). \tag{5.4}$$

Proof. Note that $\rho = (n^m)$ and that the product

$$\prod_{(i,j)\in\rho}(x_i + y_j) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j)$$

is symmetric in x and y. So, the assertion follows from (0.1).

In the special case $\lambda = (n^m)$ formula (5.4) turns into

$$s_{(n^m)}(x/y|a) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j). \tag{5.5}$$

Using definition (1.4) we derive from (5.5) the following analogue of the dual Cauchy formula which is proved in [21, formula (6.17)] and can be also deduced from the Cauchy formula for the double Schubert polynomials [17, 18].

Corollary 5.3.
$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_\lambda s_{\tilde\lambda}(x|a)s_{\lambda'}(y|-a), \tag{5.6}$$

summed over all partitions $\lambda \subset (n^m)$, where $\tilde\lambda = (n - \lambda_m, \ldots, n - \lambda_1)$ and a is an arbitrary sequence.

Proof. By (1.4),

$$s_{(n^m)}(x/y|-a^*) = \sum_{\lambda\subset(n^m)} s_{(n^m)/\lambda}(x|-a^*)s_{\lambda'}(y|-a). \tag{5.7}$$

Since $s_{(n^m)/\lambda}(x|-a^*)$ is symmetric in x, replacing x with $\tilde x = (x_m, \ldots, x_1)$ and using (1.1) we may write

$$= \sum_T \prod_{\alpha \in (n^m)/\lambda} (x_{m-T(\alpha)+1} - a_{n-T(\alpha)-c(\alpha)+1}), \tag{5.8}$$

summed over semistandard $(n^m)/\lambda$-tableaux T. Consider the bijection between the cells of the diagram $(n^m)/\lambda$ and the cells of the diagram $\tilde{\lambda}$ such that $\alpha = (i,j) \in (n^m)/\lambda$ corresponds to $\beta = (m-i+1, n-j+1) \in \tilde{\lambda}$. Obviously, the content $c(\beta)$ of the cell $\beta \in \tilde{\lambda}$ is related with $c(\alpha)$ by $c(\beta) = n - m - c(\alpha)$. Moreover, the map

$$T(\alpha) \to \tilde{T}(\beta) = m - T(\alpha) + 1$$

is a bijection between the semistandard $(n^m)/\lambda$-tableaux and the semistandard $\tilde{\lambda}$-tableaux. So, (5.8) gives

$$s_{(n^m)/\lambda}(x|-a^*) = \sum_{\tilde{T}} \prod_{\beta \in \tilde{\lambda}} (x_{\tilde{T}(\beta)} - a_{\tilde{T}(\beta)+c(\beta)}) = s_{\tilde{\lambda}}(x|a).$$

Using (5.7) and (5.5) we complete the proof.

6. Macdonald–Goulden–Greene formula

Formula (5.5) means that the polynomial $s_{(n^m)}(x/y\,|a)$ does not depend on the sequence a. It turns out, that an analogous phenomenon takes place for infinite families of variables. We shall regard the elements of the sequence a as independent variables to avoid convergence problems. Let us consider three families of variables $x = (x_i)$, $y = (y_i)$, $a = (a_i)$, $i \in \mathbb{Z}$. We define the functions $s_{\lambda/\mu}(x/y\,|a)$ in x, y and a by formula (1.5) where we allow the primed and unprimed entries of the tableaux to run through the set of all integers. This definition can be shown to be equivalent to the following formula, where the factorial Schur functions $s_{\lambda/\mu}(x|a)$ are defined by (1.1) with T running over semistandard λ/μ-tableaux with entries from $\mathbb{Z}$ (see [9] and [21]).

Proposition 6.1.

$$s_{\lambda/\mu}(x/y\,|a) = \sum_{\mu \subset \nu \subset \lambda} s_{\lambda/\nu}(x|a) s_{\nu'/\mu'}(y|-a). \tag{6.1}$$

Proof. Repeating the arguments of the proof of Proposition 1.2 we obtain that the assertion follows from the formula

$$\sum_T \prod_{\alpha \in \nu/\mu} (y_{T(\alpha)} + a_{T(\alpha)+c(\alpha)}) = s_{\nu'/\mu'}(y|-a), \tag{6.2}$$

summed over ν/μ-tableaux T with entries from $\mathbb{Z}$ whose rows strictly decrease and columns weakly decrease.

It was shown in [9] and [21] that in the case of infinite number of variables (parametrized by $\mathbb{Z}$) the factorial and supersymmetric Schur functions coincide with each other:

where the supersymmetric Schur functions $s_{\lambda/\mu}(x/y)$ are still defined by (0.2). In particular, $s_{\lambda/\mu}(x|a)$ is symmetric in a (which is not true in the finite case). Hence, we have

$$s_{\nu'/\mu'}(y|-a) = s_{\nu'/\mu'}(\widetilde{y}|-\widetilde{a}),$$

where $\widetilde{y} = (y_{-i})$ and $\widetilde{a} = (a_{-i})$. So, by (1.1),

$$s_{\nu'/\mu'}(y|-a) = \sum_{T'} \prod_{\alpha' \in \nu'/\mu'} (y_{-T'(\alpha')} + a_{-T'(\alpha')-c(\alpha')}), \qquad (6.4)$$

summed over semistandard ν'/μ'-tableaux T' with entries from $\mathbb{Z}$. Note that the map

$$T'(\alpha') \to T(\alpha) = -T'(\alpha'),$$

where $\alpha = (i,j) \in \nu/\mu$ and $\alpha' = (j,i) \in \nu'/\mu'$, is a bijection between the set of semistandard ν'/μ'-tableaux and the set of ν/μ-tableaux whose rows strictly decrease and columns weakly decrease. Obviously, $c(\alpha) = -c(\alpha')$, hence, (6.4) coincides with the left hand side of (6.2) which completes the proof.

For $a = (0)$ formula (6.1) turns into the definition of the supersymmetric Schur functions $s_{\lambda/\mu}(x/y)$. It turns out that the right hand side of (6.1) does not depend on the variables a_i and thus the Macdonald–Goulden–Greene formula (see [9] and [21]) holds for the functions $s_{\lambda/\mu}(x/y|a)$ as well.

Theorem 6.2. *One has the formula*

$$s_{\lambda/\mu}(x/y|a) = \sum_{T} \prod_{\alpha \in \lambda/\mu} (x_{T(\alpha)} + y_{T(\alpha)+c(\alpha)}), \qquad (6.5)$$

where T runs over all semistandard λ/μ-tableaux with entries from $\mathbb{Z}$.

Proof. For $a = (0)$ formula (6.5) was proved in [9] and [21]. So, it suffices to verify that $s_{\lambda/\mu}(x/y|a) = s_{\lambda/\mu}(x/y)$. Using (6.1) and (6.3) we obtain

$$s_{\lambda/\mu}(x/y|a) = \sum_{\nu} s_{\lambda/\nu}(x/-a)s_{\nu'/\mu'}(y/a). \qquad (6.6)$$

By the symmetry property (0.3) we have $s_{\nu'/\mu'}(y/a) = s_{\nu/\mu}(a/y)$. Hence, using the definition of the supersymmetric Schur functions we can rewrite (6.6) as follows

$$s_{\lambda/\mu}(x/y|a) = \sum_{\nu,\rho,\sigma} s_{\lambda/\rho}(x)s_{\rho'/\nu'}(-a)s_{\nu/\sigma}(a)s_{\sigma'/\mu'}(y)$$

$$= \sum_{\rho,\sigma} s_{\lambda/\rho}(x)s_{\sigma'/\mu'}(y)s_{\rho'/\sigma'}(-a/a). \qquad (6.7)$$

Note now that $s_{\rho'/\sigma'}(-a/a) = 0$ unless $\rho = \sigma$. Indeed, suppose that there exists a cell $\alpha_0 \in \rho'/\sigma'$. Then, applying (6.5) with $a = (0)$ we get

$$s_{\rho'/\sigma'}(-a/a) = s_{\rho'/\sigma'}(-a/\tau^{-c(\alpha_0)}a)$$

$$= \sum \prod (-a_{T(\alpha)} + a_{T(\alpha)+c(\alpha)-c(\alpha_0)}) = 0.$$

Thus, (6.7) coincides with $s_{\lambda/\mu}(x/y)$ which completes the proof.

7. Shifted supersymmetric Schur polynomials and a basis in the center of $U(\mathfrak{gl}(m|n))$

We need to introduce super-analogues of the shifted symmetric polynomials (cf. [29, 31, 32]). A polynomial in two families of variables $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$ will be called *shifted supersymmetric* if it is supersymmetric in the variables

$$(u_1 + m - 1, \ u_2 + m - 2, \ \ldots, \ u_m) \qquad \text{and} \qquad (v_1, \ v_2 - 1, \ \ldots, \ v_n - n + 1).$$

We shall denote the algebra of shifted supersymmetric polynomials by $\Lambda^*(m|n)$. It follows from its definition that $\Lambda^*(m|n)$ is isomorphic to the algebra of supersymmetric polynomials in x and y.

Let us consider the polynomials $s_{\lambda/\mu}(x/y\,|a)$ with the sequence a defined by $a_i = -m + i$ and put in $s_{\lambda/\mu}(x/y\,|a)$

$$x_i = u_{m-i+1} - m + i \qquad \text{for} \quad i = 1, \ldots, m,$$
$$y_j = v_j + m - j \qquad \text{for} \quad j = 1, \ldots, n.$$

Then we obtain a shifted supersymmetric polynomial in u and v which will be denoted by $s^*_{\lambda/\mu}(u/v)$ and will be called *shifted supersymmetric Schur polynomial*. In the case $n = 0$ it coincides with the shifted Schur polynomial $s^*_{\lambda/\mu}(u)$ (see [29, 31, 32]) which can be defined by the formula

$$s^*_{\lambda/\mu}(u) = \sum_T \prod_{\alpha \in \lambda/\mu} (u_{T(\alpha)} - c(\alpha)), \tag{7.1}$$

summed over λ/μ-tableaux T with entries in $\{1, \ldots, m\}$ whose rows weakly decrease and columns strictly decrease. The highest component of $s^*_{\lambda/\mu}(u)$ is the usual skew Schur polynomial $s_{\lambda/\mu}(u)$.

We shall state now some properties of the polynomials $s^*_{\lambda/\mu}(u/v)$, which can be easily derived from the corresponding properties of the polynomials $s_{\lambda/\mu}(x/y\,|a)$.

First we give a combinatorial interpretation of $s^*_{\lambda/\mu}(u/v)$.

To distinguish the indices of u and v let us identify the indices of v with the symbols $1', \ldots, n'$. Consider the diagram of shape λ/μ and fill it with the indices $1', \ldots, n', 1, \ldots, m$ such that:

(a) In each row (resp. column) each primed index is to the left (resp. above) from each unprimed index.

(b) Primed indices strictly decrease along rows and weakly decrease down columns.

(c) Unprimed indices weakly decrease along rows and strictly decrease down columns.

Denote the resulting tableau by T

Proposition 7.1. *One has the formula*

$$s^*_{\lambda/\mu}(u/v) = \sum_T \prod_{\substack{\alpha \in \lambda/\mu \\ T(\alpha)\,\text{unprimed}}} (u_{T(\alpha)} - c(\alpha)) \prod_{\substack{\alpha \in \lambda/\mu \\ T(\alpha)\,\text{primed}}} (v_{T(\alpha)} + c(\alpha)). \qquad (7.2)$$

Proof. This follows immediately from Proposition 1.2. It suffices to use (1.5) with x replaced by $\widetilde{x} = (x_m, \ldots, x_1)$.

In particular, for the *elementary* and *complete shifted supersymmetric polynomials*

$$e^*_k(u/v) := s^*_{(1^k)}(u/v) \quad \text{and} \quad h^*_k(u/v) := s^*_{(k)}(u/v) \quad \text{we have}$$

$$e^*_k(u/v) = \sum_{p+q=k} \sum_{\substack{i_1 > \cdots > i_p \\ j_1 \geq \cdots \geq j_q}} v_{j_1}(v_{j_2} - 1) \cdots (v_{j_q} - q + 1)(u_{i_1} + q) \cdots (u_{i_p} + k - 1),$$

$$h^*_k(u/v) = \sum_{p+q=k} \sum_{\substack{i_1 \geq \cdots \geq i_p \\ j_1 > \cdots > j_q}} v_{j_1}(v_{j_2} + 1) \cdots (v_{j_q} + q - 1)(u_{i_1} - q) \cdots (u_{i_p} - k + 1).$$

Using (7.1) we can rewrite (7.2) in the following equivalent form.

Corollary 7.2. *One has*

$$s^*_{\lambda/\mu}(u/v) = \sum_{\mu \subset \nu \subset \lambda} s^*_{\lambda/\nu}(u)\, s^*_{\nu'/\mu'}(v). \qquad (7.3)$$

This implies that the highest component of $s^*_{\lambda/\mu}(u/v)$ is the supersymmetric Schur polynomial $s_{\lambda/\mu}(u/v)$. So, the polynomials $s^*_\lambda(u/v)$ with $\lambda \subset (m,n)$-hook form a basis in $\Lambda^*(m|n)$.

The following are reformulations of Theorems 4.4 and 4.5′ for the shifted supersymmetric polynomials; cf. [29, 31] (we use the notation from Section 4).

Theorem 7.3. *Let* λ, ζ *be partitions which are contained in the* (m,n)*-hook.*
  (i) *If* $\lambda \not\subset \zeta$ *then*

$$s^*_\lambda(\xi/\eta) = 0. \qquad (7.4)$$

  (ii) *If* $\lambda = \zeta$ *then*

$$s^*_\lambda(\xi/\eta) = H(\lambda), \qquad (7.5)$$

where $H(\lambda)$ *is the product of the hook lengths of all cells of* λ*.*

Theorem 7.4. *Let* $f(u/v)$ *be a shifted supersymmetric polynomial such that*

$$f(u/v) = s_\lambda(u/v) + \text{lower terms}$$

for some partition $\lambda \subset (m,n)$*-hook, and* $f(\xi/\eta) = 0$ *for any partition* $\zeta \subset (m,n)$*-hook with* $|\zeta| < |\lambda|$*. Then* $f(u/v) = s^*_\lambda(u/v)$*.*

A distinguished linear basis in the center of the universal enveloping algebra

weight representation is a shifted Schur polynomial $s_\lambda^*(u)$. It turns out, that this construction can be easily carried to the case of the Lie superalgebra $\mathfrak{gl}(m|n)$. Below we formulate the corresponding theorem and briefly outline its proof. Another approach to this construction, based on the super-analogues of the higher Capelli identities is contained in Section 8.

We denote by E_{ij}, $i,j = 1, \ldots, m+n$ the standard basis of the Lie superalgebra $\mathfrak{gl}(m|n)$. The $\mathbb{Z}_2$-grading on $\mathfrak{gl}(m|n)$ is defined by $E_{ij} \mapsto p(i)+p(j)$, where $p(i) = 0$ or 1 depending on whether $i \leq m$ or $i > m$. The commutation relations in this basis are given by

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}(-1)^{(p(i)+p(j))(p(k)+p(l))}. \tag{7.6}$$

Given $w = (u_1, \ldots, u_m, v_1, \ldots, v_n) \in \mathbb{C}^{m+n}$ we consider an arbitrary highest weight $\mathfrak{gl}(m|n)$-module $L(w)$ with the highest weight w. That is, $L(w)$ is generated by a nonzero vector ψ such that

$$
\begin{aligned}
E_{ii}\,\psi = u_i\,\psi && \text{for} \quad i = 1, \ldots, m, \\
E_{m+j,m+j}\,\psi = v_j\,\psi && \text{for} \quad j = 1, \ldots, n, \\
E_{ij}\,\psi = 0 && \text{for} \quad 1 \leq i < j \leq m+n.
\end{aligned}
$$

Every element z of the center $\mathrm{Z}(\mathfrak{gl}(m|n))$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n))$ acts in $L(w)$ as a scalar $\chi(z)$. For a fixed z the scalar $\chi(z)$ is a shifted supersymmetric polynomial in u and v and the map $z \mapsto \chi(z)$ defines an algebra isomorphism

$$\chi : \mathrm{Z}(\mathfrak{gl}(m|n)) \to \Lambda^*(m|n), \tag{7.7}$$

which is called the Harish-Chandra isomorphism; see [16, 38, 40].

Our aim now is to give an explicit description of the basis of the algebra $\mathrm{Z}(\mathfrak{gl}(m|n))$ formed by the preimages $\chi^{-1}(s_\lambda^*(u/v))$ of the basis elements of $\Lambda^*(m|n)$. Let us introduce some more notations.

We shall need to consider matrices with entries from superalgebras. All our matrices will be even. That is, if $B = (B_{ia})$ is a $(m,n) \times (m',n')$-matrix whose entries are homogeneous entries of a superalgebra $\mathcal{B}$, we shall always have $p(B_{ia}) = p(i)+p(a)$, where $p(a) = 0$ or 1 depending on whether $a \leq m'$ or $a > m'$. A matrix B will be identified with an element of the tensor product

$$B = \sum_{i,a} e_{ia} \otimes B_{ia}(-1)^{p(a)(p(i)+1)} \in \mathrm{Mat}_{(m,n)\times(m',n')} \otimes \mathcal{B},$$

where the e_{ia} are the standard matrix units.

More generally, given k matrices $B^{(1)}, \ldots, B^{(k)}$ of the size $(m,n) \times (m',n')$ we define their tensor product $B^{(1)} \otimes \cdots \otimes B^{(k)}$ as an element

$$\sum e_{i_1 a_1} \otimes \cdots \otimes e_{i_k a_k} \otimes B^{(1)}_{i_1 a_1} \cdots B^{(k)}_{i_k a_k} (-1)^{\gamma(I,A)} \in (\mathrm{Mat}_{(m,n)\times(m',n')})^{\otimes k} \otimes \mathcal{B},$$

where

$$\gamma(I,A) = \sum^k p(a_r)(p(i_r)+1) + \sum_{} (p(i_r)+p(a_r))(p(i_s)+p(a_s)).$$

The supertrace of an element

$$B = \sum e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes B_{i_1,\ldots,i_k;j_1,\ldots,j_k} \in (\mathrm{Mat}_{(m,n)\times(m,n)})^{\otimes k} \otimes \mathcal{B}$$

is defined by

$$\mathrm{str}\, B = \sum_{i_1,\ldots,i_k} B_{i_1,\ldots,i_k;i_1,\ldots,i_k}(-1)^{p(i_1)+\cdots+p(i_k)}.$$

Using the natural action of the symmetric group S_k in the space $(\mathbb{C}^{m|n})^{\otimes k}$ we represent each element of S_k by a linear combination of tensor products of matrices. In particular, the transposition $(i,j) \in S_k$, $i < j$, corresponds to the element

$$P_{ij} = \sum_{a,b} 1 \otimes \cdots \otimes 1 \otimes e_{ab} \otimes 1 \otimes \cdots \otimes 1 \otimes e_{ba} \otimes 1 \otimes \cdots \otimes 1 \, (-1)^{p(b)},$$

where the tensor factors e_{ab} and e_{ba} take the ith and jth places, respectively.

We can now describe the construction of a basis in $Z(\mathfrak{gl}(m|n))$.

Set $\widehat{E}_{ij} = E_{ij}(-1)^{p(j)}$ and denote by $\widehat{E}$ the $(m,n) \times (m,n)$-matrix whose ijth entry is $\widehat{E}_{ij}$.

Following [29], for a partition λ and a standard λ-tableau T we denote by v_T the corresponding vector of the Young orthonormal basis with respect to an invariant inner product $(\ ,\ )$ in the irreducible representation V^λ of the symmetric group S_k, $k = |\lambda|$. We let $c_T(r) = j - i$ if the cell $(i,j) \in \lambda$ is occupied by the entry r of the tableau T. Given two standard λ-tableaux T and T', introduce the matrix element

$$\Psi_{TT'} = \sum_{s \in S_k} (s \cdot v_T, v_{T'}) \cdot s^{-1} \in \mathbb{C}[S_k]. \qquad (7.8)$$

Theorem 7.5. *The element*

$$\mathbb{S}_\lambda = \frac{1}{H(\lambda)} \mathrm{str}\, (\widehat{E} - c_T(1)) \otimes \cdots \otimes (\widehat{E} - c_T(k)) \cdot \Psi_{TT}, \qquad (7.9)$$

is independent of a λ-tableau T. The set of elements $\mathbb{S}_\lambda$ with $\lambda \subset (m,n)$-hook forms a basis in $Z(\mathfrak{gl}(m|n))$. Moreover, the image of $\mathbb{S}_\lambda$ under the Harish-Chandra isomorphism is $s_\lambda^*(u/v)$.

Outline of the proof. In the case $n = 0$ this theorem was proved in [29]. It constitutes the 'difficult part' of the proof of the higher Capelli identities. A straightforward generalization of those arguments proves Theorem 7.5. For this one uses the following R-matrix form of the defining relations in $U(\mathfrak{gl}(m|n))$ (cf. [27]):

$$R(u-v) \cdot \widehat{E}(u) \otimes \widehat{E}(v) = \widehat{E}(v) \otimes \widehat{E}(u) \cdot R(u-v),$$

where $R(u) = 1 + P_{12}\, u$.

Examples. For the partitions of weight ≤ 2 we have

$$\mathbb{S}_{(1)} = \mathrm{str}\, \widehat{E} = \sum E_{ii},$$

$$\mathbb{S}_{(2)} = \frac{1}{2} \operatorname{str} \left(\widehat{E} \otimes (\widehat{E} - 1) \cdot (1 + P_1) \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(E_{ii}(E_{jj} - (-1)^{p(j)}) + E_{ij}(E_{ji} - \delta_{ji}(-1)^{p(i)})(-1)^{p(j)} \right),$$

$$\mathbb{S}_{(1^2)} = \frac{1}{2} \operatorname{str} \left(\widehat{E} \otimes (\widehat{E} + 1) \cdot (1 - P_1) \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(E_{ii}(E_{jj} + (-1)^{p(j)}) - E_{ij}(E_{ji} + \delta_{ji}(-1)^{p(i)})(-1)^{p(j)} \right).$$

8. Super Capelli identities

Here we formulate a super-analogue of the higher Capelli identities obtained in [28–30].

Consider the supercommutative algebra $\mathcal{Z}$ with generators z_{ia} where $i = 1, \ldots, m+$■ n and $a = 1, \ldots, m' + n'$ and the $\mathbb{Z}_2$-grading given by $z_{ia} \mapsto p(i) + p(a)$. Define the representation π of the Lie superalgebra $\mathfrak{gl}(m|n)$ in $\mathcal{Z}$ by

$$\pi(E_{ij}) = \sum_{a=1}^{m'+n'} z_{ia}\partial_{ja},$$

where $\partial_{ja} = \partial/\partial z_{ja}$ is the left derivation. This definition can be rewritten in the matrix form as follows:

$$\pi(\widehat{E}) = Z\,D', \tag{8.1}$$

where Z is the $(m, n) \times (m', n')$-matrix (z_{ia}) and and D' is the $(m', n') \times (m, n)$-matrix (∂'_{ai}) with $\partial'_{ai} = \partial_{ia}(-1)^{p(i)}$.

For a partition λ with $|\lambda| = k$ denote by χ^λ the irreducible character of S_k. We identify χ^λ with an element of the group algebra $\mathbb{C}[S_k]$:

$$\chi^\lambda = \sum_{s \in S_k} \chi^\lambda(s) \cdot s.$$

Define the differential operator Δ_λ by

$$\Delta_\lambda = \frac{1}{k!} \operatorname{str} \left(Z^{\otimes k} \cdot D'^{\otimes k} \cdot \chi^\lambda \right).$$

The following is a super-analogue of the higher Capelli identities (cf. [28–30]).

Theorem 8.1.

$$\pi(\mathbb{S}_\lambda) = \Delta_\lambda. \tag{8.2}$$

We outline two proofs of this identity. The first proof is based on the properties of the shifted supersymmetric polynomials. The second one uses a super-analogue

The first proof. We follow again the corresponding arguments from [29]. First, one verifies that the operator Δ_λ commutes with both the actions of the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{gl}(m'|n')$ in $\mathcal{Z}$; the latter is given by

$$\pi'(E'_{ab}) = \sum_{i=1}^{m+n} z_{ia}\partial_{ib}(-1)^{(p(a)+p(b))p(i)},$$

where the E'_{ab} are the standard generators of $\mathfrak{gl}(m'|n')$. This implies that Δ_λ is the image of a certain element $\mathbb{S}'_\lambda \in \mathrm{Z}(\mathfrak{gl}(m|n))$ under π (cf. [12, 27]). To prove that $\mathbb{S}_\lambda = \mathbb{S}'_\lambda$ we compare their images under the Harish-Chandra isomorphism. Set $s'_\lambda(u/v) = \chi(\mathbb{S}'_\lambda)$. By Theorem 7.5, $\chi(\mathbb{S}_\lambda) = s^*_\lambda(u/v)$. We use Theorem 7.4 to prove that $s'_\lambda(u/v) = s^*_\lambda(u/v)$. Using (8.1) we check that both sides of (8.2) agree modulo lower terms (cf. [29, 31]). This proves that the polynomials $s'_\lambda(u/v)$ and $s^*_\lambda(u/v)$ have the same highest component which coincides with the supersymmetric Schur polynomial $s_\lambda(u/v)$.

By Theorem 7.3, to complete the proof we have to verify that $s'_\lambda(\xi/\eta)$ is zero for any partition $\zeta \subset (m,n)$-hook such that $|\zeta| < |\lambda|$. Let us consider the superalgebra $\mathcal{Z}$ with the parameters m' and n' being sufficiently large, so that $m' \geq \max\{m,|\zeta|\}$ and $n' \geq \max\{n,|\zeta|\}$.

Introduce the following element of $\mathcal{Z}$:

$$\psi_\zeta = \Delta_1^{\eta_1-\eta_2}\Delta_2^{\eta_2-\eta_3}\cdots\Delta_n^{\eta_n}\prod_{(i,j)\in\xi} z_{i,m'+j},$$

where $\Delta_r = \det[z_{m+i,m'+j}]_{1\leq i,j\leq r}$ and the product is taken in any fixed order. It can be easily checked that ψ_ζ satisfies the relations

$$\begin{aligned}
\pi(E_{ii})\,\psi_\zeta &= \xi_i\,\psi_\zeta & \text{for} \quad i &= 1,\ldots,m,\\
\pi(E_{m+j,m+j})\,\psi_\zeta &= \eta_j\,\psi_\zeta & \text{for} \quad j &= 1,\ldots,n,\\
\pi(E_{ij})\,\psi_\zeta &= 0 & \text{for} \quad 1 &\leq i < j \leq m+n.
\end{aligned}$$

This means that ψ_ζ generates a $\mathfrak{gl}(m|n)$-module with the highest weight (ξ,η). Hence, ψ_ζ is an eigenvector for the operator Δ_λ with the eigenvalue $s'_\lambda(\xi/\eta)$. However, the degree of ψ_ζ equals $|\zeta|$ and so, if $|\zeta| < |\lambda| = k$ then ψ_ζ is annihilated by Δ_λ, that is, $s'_\lambda(\xi/\eta) = 0$ which completes the proof.

Theorem 8.2. *Let T and T' be two standard tableaux of the shape λ. Then*

$$\pi\left(\left(\widehat{E} - c_T(1)\right) \otimes \cdots \otimes \left(\widehat{E} - c_T(k)\right)\cdot\Psi_{TT'}\right) = Z^{\otimes k}\cdot(D')^{\otimes k}\cdot\Psi_{TT'}. \tag{8.3}$$

Proof. Following [30], we use some properties of the Jucys–Murphy elements in the group algebra for the symmetric group. However, the arguments from [30] can be modified to avoid using the Wick formula and the Olshanskiĭ special symmetrization map.

We use induction on k. Denote by U the tableau obtained from T by removing the cell with the entry k. From the branching property of the Young basis $\{v_T\}$ one can easily derive that

where 'const' is a nonzero constant (more precisely, const $= \dim \mu/(k-1)!$ where μ is the shape of U and $\dim \mu = \dim V^\mu$).

So, we can rewrite the left hand side of (8.3) as follows:

$$\text{const} \cdot (ZD' - c_T(1)) \otimes \cdots \otimes (ZD' - c_T(k-1)) \cdot \Psi_{UU} \otimes (ZD' - c_T(k)) \cdot \Psi_{TT'}.$$

By the induction hypothesis, this equals

$$\text{const} \cdot Z^{\otimes k-1} \cdot (D')^{\otimes k-1} \cdot \Psi_{UU} \otimes (ZD' - c_T(k)) \cdot \Psi_{TT'}$$

$$= Z^{\otimes k-1} \cdot (D')^{\otimes k-1} \otimes (ZD' - c_T(k)) \cdot \Psi_{TT'}$$

$$= \left(\sum e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes z_{i_1 a_1} \cdots z_{i_{k-1} a_{k-1}} \partial'_{a_1 j_1} \cdots \partial'_{a_{k-1} j_{k-1}} \right.$$

$$\left. \left(\sum z_{i_k a_k} \partial'_{a_k j_k} - \delta_{i_k j_k} c_T(k) \right) \right) (-1)^{\alpha(I,J,A)} \cdot \Psi_{TT'},$$

where

$$\alpha(I, J, A) = \sum_{r=1}^{k} p(j_r)(p(i_r) + 1) + \sum_{1 \le r < s \le k-1} (p(a_r) + p(j_r))(p(i_s) + p(a_s))$$

$$+ \sum_{1 \le r < s \le k} (p(i_r) + p(j_r))(p(i_s) + p(j_s)).$$

Now we transform this expression using the relations

$$\partial'_{bj} z_{ia} = z_{ia} \partial'_{bj} (-1)^{(p(i)+p(a))(p(j)+p(b))} + \delta_{ab} \delta_{ij} (-1)^{p(j)}$$

to obtain

$$\left(\sum e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes z_{i_1 a_1} \cdots z_{i_k a_k} \partial'_{a_1 j_1} \cdots \partial'_{a_k j_k} (-1)^{\beta_k(I,J,A)} \right) \cdot \Psi_{TT'}$$

$$+ \left(\sum e_{i_1 j_1} \otimes \cdots \otimes e_{i_{k-1} j_{k-1}} \otimes 1 \otimes z_{i_1 a_1} \cdots z_{i_{k-1} a_{k-1}} \partial'_{a_1 j_1} \cdots \partial'_{a_{k-1} j_{k-1}} (-1)^{\beta_{k-1}(I,J,A)} \right) \Bigg|$$

$$\times (P_{1k} + \cdots + P_{k-1,k} - c_T(k)) \cdot \Psi_{TT'}, \tag{8.4}$$

where

$$\beta_k(I, J, A) = \sum_{r=1}^{k} p(j_r)(p(i_r) + 1) + \sum_{1 \le r < s \le k} (p(a_r) + p(j_r))(p(i_s) + p(a_s))$$

$$+ \sum_{1 \le r < s \le k} (p(i_r) + p(j_r))(p(i_s) + p(j_s)).$$

Note that $P_{1k} + \cdots + P_{k-1,k}$ is the image of the Jucys–Murphy element $(1k) + \cdots + (k-1, k) \in \mathbb{C}[S_k]$ (see [14] and [26]). It has the property

$$((1k) + \cdots + (k-1, k)) \cdot \Psi_{TT'} = c_T(k) \cdot \Psi_{TT'},$$

which was also used in [30] and can be easily derived from the following formula due to Jucys and Murphy:

This proves that the second summand in (8.4) is zero, while the first one coincides with the right hand side of (8.3). Theorem 8.2 is proved.

The second proof of Theorem 8.1. Put $T = T'$ in Theorem 8.2 and take the supertrace of both sides of (8.3). On the left hand side we get $H(\lambda) \pi(\mathbb{S}_\lambda)$ while for the right hand side we have

$$\operatorname{str} Z^{\otimes k} \cdot (D')^{\otimes k} \cdot \Psi_{TT} = \frac{1}{k!} \sum_{s \in S_k} \operatorname{str} s \cdot Z^{\otimes k} \cdot (D')^{\otimes k} \cdot \Psi_{TT} \cdot s^{-1}$$

$$= \frac{1}{\dim \lambda} \operatorname{str} Z^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\lambda.$$

Here we have used the invariance of $Z^{\otimes k}$ and $(D')^{\otimes k}$ under the conjugations by elements $s \in S_k$ and the following equality of elements of the group algebra of S_k:

$$\frac{1}{k!} \sum_{s \in S_k} s \cdot \Psi_{TT} \cdot s^{-1} = \frac{1}{\dim \lambda} \chi^\lambda.$$

So, on the right hand side we get $H(\lambda) \Delta_\lambda$ which completes the proof.

Example. In the case of $\lambda = (1^k)$ the identity (8.2) was obtained by M. Nazarov [27] in another form. Namely, a formal series $B(t)$ whose coefficients are generators of $Z(\mathfrak{gl}(m|n))$ was constructed in [27] with the use of some properties of the Yangian for the Lie superalgebra $\mathfrak{gl}(m|n)$. The explicit expression for $B(t)$ has the form of a 'quantum' analogue of Berezinian:

$$B(t) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(1 + \frac{\widehat{E}}{t}\right)_{\sigma(1),1} \cdots \left(1 + \frac{\widehat{E}}{t-m+1}\right)_{\sigma(m),m}$$

$$\times \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \left(1 + \frac{\widehat{E}}{t-m+1}\right)^*_{m+\tau(1),m+1} \cdots \left(1 + \frac{\widehat{E}}{t-m+n}\right)^*_{m+\tau(n),m+n},$$

(8.5)

where $A^* = (A^{-1})^{st}$ and st is the matrix supertransposition: $(B^{st})_{ij} = B_{ji}(-1)^{p(i)(p(j)+1)}$. The image of $B(t)$ under the Harish-Chandra isomorphism coincides with its eigenvalue on the highest vector ψ of the highest weight $\mathfrak{gl}(m|n)$-module $L(w)$, $w = (u,v) \in \mathbb{C}^{m|n}$. The eigenvalue of the first determinant in (8.5) on ψ is

$$\frac{(t+u_1) \cdots (t+u_m - m + 1)}{t(t-1) \cdots (t-m+1)}.$$

To find the eigenvalue of the second determinant we may replace the matrix $\widehat{E}$ with its submatrix $\widetilde{E} = (\widehat{E}_{ij})_{m+1 \leq i,j \leq m+n}$. However, the determinant

$$\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \left(1 + \frac{\widetilde{E}}{t-m+1}\right)^*_{m+\tau(1),m+1} \cdots \left(1 + \frac{\widetilde{E}}{t-m+n}\right)^*_{m+\tau(n),m+n}$$

equals

$$\left(\sum_{\tau} \operatorname{sgn}(\tau) \left(1 + \frac{\widetilde{E}}{t-m+1}\right) \cdots \left(1 + \frac{\widetilde{E}}{t-m+n}\right)\right)^{-1},$$

which follows from [27, Proposition 3] and can be also proved directly by using an R-matrix form of the defining relations in $U(\mathfrak{gl}(n))$; see, e.g., [25]. So, its eigenvalue on ψ is

$$\frac{(t - m + 1) \cdots (t - m + n)}{(t - v_1 - m + 1) \cdots (t - v_n - m + n)}.$$

Thus,

$$\chi(B(t)) = \frac{(t + u_1) \cdots (t + u_m - m + 1)(t - m + 1) \cdots (t - m + n)}{t(t - 1) \cdots (t - m + 1)(t - v_1 - m + 1) \cdots (t - v_n - m + n)}.$$

Relation (2.5) implies that

$$\chi(B(t)) = 1 + \sum_{k=1}^{\infty} \frac{e_k^*(u/v)}{t(t - 1) \cdots (t - k + 1)}.$$

By Theorem 7.5, $\chi(\mathbb{S}_{(1^k)}) = e_k^*(u/v)$, hence,

$$B(t) = 1 + \sum_{k=1}^{\infty} \frac{\mathbb{S}_{(1^k)}}{t(t - 1) \cdots (t - k + 1)}.$$

Using (8.2) we get the identity (see [27]):

$$\pi(B(t)) = 1 + \sum_{k=1}^{\infty} \frac{\Delta_{(1^k)}}{t(t - 1) \cdots (t - k + 1)}.$$

For $n = n' = 0$ it turns into the classical Capelli identity; see, e.g., [11, 12].

References

[1] A. Abderrezzak, Généralisation d'identités de Carlitz, Howard et Lehmer, *Aequat. Math.* **49** (1995), 36–46.

[2] A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* **64** (1987), 118–175.

[3] N. Bergeron and A. M. Garsia, Sergeev's formula and the Littlewood–Richardson rule, *Linear and Multilinear Algebra* **27** (1990), 79–100.

[4] L. C. Biedenharn and J. D. Louck, A new class of symmetric polynomials defined in terms of tableaux, *Advances in Appl. Math.* **10** (1989), 396–438.

[5] L. C. Biedenharn and J. D. Louck, Inhomogeneous basis set of symmetric polynomials defined by tableaux, *Proc. Nat. Acad. Sci. U.S.A.* **87** (1990), 1441–1445.

[6] W. Y. C. Chen and J. D. Louck, The factorial Schur function, *J. Math. Phys.* **34** (1993), 4144–4160.

[7] P. H. Dondi and P. D. Jarvis, Diagram and superfield techniques in the classical superalgebras, *J. Phys. A* **14** (1981), 547–563.

[8] I. M. Gessel and G. X. Viennot, Binomial determinants, paths, and hook

[9] I. Goulden and C. Greene, A new tableau representation for supersymmetric Schur functions, *J. Algebra.* **170** (1994), 687–703.

[10] I. P. Goulden and A. M. Hamel, Shift operators and factorial symmetric functions, University of Waterloo Preprint CORR 92–10, April 1, 1992, to appear in *J. Comb. Theor. A.*

[11] R. Howe, Remarks on classical invariant theory, *Trans. AMS* **313** (1989), 539–570.

[12] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* **290** (1991), 569–619.

[13] J. van der Jeugt, J. W. B. Hughes, R. C. King and J. Thierry-Mieg, Character formulae for irreducible modules of the Lie superalgebra $sl(m|n)$, *J. Math. Phys.* **31** (1990), 2278–2304.

[14] A.-A. A. Jucys, Symmetric polynomials and the center of the symmetric group ring, *Reports Math. Phys.* **5** (1974), 107–112.

[15] V. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.

[16] V. G. Kac, Representations of classical Lie superalgebras, *in* "Differential Geometry Methods in Mathematical Physics II", (K. Bleuer, H. R. Petry, A. Reetz, Eds.), Lecture Notes in Math., Vol. 676, pp. 597–626. Springer-Verlag, Berlin/Heidelberg/New York, 1978.

[17] A. Lascoux, Letter to the author.

[18] A. Lascoux, Classes de Chern des variétés de drapeaux, *Comptes Rendus Acad. Sci. Paris, Sér. I* **295** (1982), 393–398.

[19] D. E. Littlewood, "The theory of group characters and matrix representations of groups", 2nd edition, Clarendon Press, Oxford, 1950.

[20] I. G. Macdonald, "Symmetric functions and Hall polynomials", 2nd edition, Oxford University Press, 1995.

[21] I. G. Macdonald, Schur functions: theme and variations, *in* "Actes 28-e Séminaire Lotharingien", pp. 5–39. Publ. I.R.M.A. Strasbourg, 1992, 498/S–27.

[22] I. G. Macdonald, "Notes on Schubert polynomials", Publ. LACIM, Univ. du Québec à Montréal, 1991.

[23] N. Metropolis, G. Nicoletti, G. C. Rota, A new class of symmetric functions, *in* "Mathematical Analysis and Applications," pp. 563–575. Advances in Mathematics Supplementary Studies, Vol. **7B**, Academic Press, New York, 1981.

[24] A. Molev, M. Nazarov, Capelli identities for classical groups, Preprint University of Wales 95-21, Swansea, 1995.

[25] A. I. Molev, M. L. Nazarov and G. I. Olshanskiĭ, Yangians and classical Lie algebras, *Uspekhi Matem. Nauk* **51**, (1996) no. 2, 27–104.

[26] G. E. Murphy, A new construction of Young's seminormal representation of the symmetric group, *J. Algebra* **69** (1981), 287–291.

[27] M. L. Nazarov, Quantum Berezinian and the classical Capelli identity, *Lett. Math. Phys.* **21** (1991), 123–131.

[28] M. L. Nazarov, Yangians and Capelli identities, Preprint, 1995, q-alg/9601027.

[29] A. Okounkov, Quantum immanants and higher Capelli identities, Preprint, 1995, q-alg/9602028.

[30] A. Okounkov, Young basis, Wick formula and higher Capelli identities, Preprint, 1995, q-alg/9602027.

[31] A. Okounkov and G. Olshanskiĭ, Shifted Schur functions, Preprint, 1995, q-alg/9605042.

[32] G. I. Olshanskiĭ, Quasi-symmetric functions and factorial Schur functions, Preprint, 1995.

[33] P. Pragacz, Algebro-geometric applications of Schur S- and Q-polynomials, *in* "Topics in Invariant Theory," Seminaire d'Algebre Paul Dubriel et Marie-Paule Malliavin, Proceedings, Lecture Notes in Math., Vol. 1478, pp. 130–191. Springer-Verlag, New York/Berlin, 1991.

[34] P. Pragacz and A. Thorup, On a Jacobi–Trudi identity for supersymmetric polynomials, *Adv. Math.* **95** (1992), 8–17.

[35] J. B. Remmel, The combinatorics of (k, l)-hook Schur functions, *Contemp. Math.* **34** (1984), 253–287.

[36] B. Sagan, "The symmetric group: representations, combinatorial algorithms, and symmetric functions", Wadsworth & Brooks / Cole mathematics series, Pacific Grove, CA, 1991.

[37] S. Sahi, The spectrum of certain invariant differential operators associated to a Hermitian symmetric space, *in* "Lie theory and geometry", Progr. Math. **123**, Boston: Birkhauser 1994, 569–576.

[38] M. Scheunert, Casimir elements of Lie superalgebras, *in* "Differential Geometry Methods in Mathematical Physics", pp. 115–124, Reidel, Dordrecht, 1984.

[39] R. P. Stanley, Unimodality and Lie superalgebras, *Stud. Appl. Math.* **72** (1985), 263–281.

[40] J. R. Stembridge, A characterization of supersymmetric polynomials, *J. Algebra* **95** (1985), 439–444.